$f(x) = \frac{1}{\sqrt{2\pi}} e$

http://www.les-mathematiques.net/phorum/read.php?12,1821574

lois gamma liées à la loi normale.

• $\Gamma(k = \frac{1}{2}, \lambda = \frac{1}{2})$ $\quad f(x) = \frac{1}{\sqrt{x}} e^{-x/2} \frac{1}{\sqrt{2\pi}}$

car si X suit $N(0,1)$ alors $Y = X^2$ vérifie $F_Y(x) = 2F_X(\sqrt{x}) - 1$

$$\Leftrightarrow f_Y(x) = \frac{1}{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-x/2}$$

donc $Y = X^2$ suit $\Gamma(k = \frac{1}{2}, \lambda = \frac{1}{2})$

• $\Gamma(k = \frac{n}{2}, \lambda = \frac{1}{2})$ $\quad f(x) = \frac{x^{\frac{n}{2}-1} e^{-x/2}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})}$

si les X_i indépendantes suivent $N(0,1)$ alors

$$S_n = X_1^2 + \dots + X_n^2 \quad \text{suit} \quad \Gamma(\tfrac{n}{2}, \tfrac{1}{2})$$

preuve par récurrence :

$$f_{S_{n+1}}(x) = \int_0^x \frac{t^{\frac{n}{2}-1} e^{-t/2}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \frac{(x-t)^{-\frac{1}{2}} e^{-(x-t)/2}}{2^{\frac{1}{2}} \Gamma(\frac{1}{2})}$$

$$= \frac{e^{-x/2}}{2^{\frac{n+1}{2}}} \times \frac{1}{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})} \int_0^x t^{\frac{n}{2}-1} (x-t)^{-\frac{1}{2}} dt$$

reste à trouver que $\underbrace{\int_0^x t^{\frac{n}{2}-1} (x-t)^{-\frac{1}{2}}}_{I_n} dt = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})} \times x^{\frac{n}{2}-\frac{1}{2}}$

• si $n = 2k$ $\quad I_n = \underbrace{(k-1)!}_{\Gamma(\frac{n}{2})} \underbrace{\frac{\left[(x-t)^{\frac{2k-1}{2}}\right]}{\frac{1}{2} \times \dots \times \frac{2k-1}{2}}}_{\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}} \to x^{\frac{n}{2}-\frac{1}{2}}$ $\quad$ OK

• si $n = 2k+1$ $\quad \int_0^x t^{\frac{n}{2}-1} (x-t)^{-\frac{1}{2}} dt = \underbrace{x^{\frac{n-1}{2}}}_{\text{OK}} \int_0^1 u^{k-1} (1-u)^{-\frac{1}{2}} du$

$$= x^{\frac{n-1}{2}} \underbrace{\int_0^{\pi/2} 2 \sin^{2k} \theta \, d\theta}_{I_k}$$

$$I_k = \int_0^{\pi/2} 2 \left(\frac{e^{i\theta} + e^{-i\theta}}{2i}\right)^{2k} d\theta = \text{Newton} = \frac{2}{(2i)^{2k}} \left[\sum_{\substack{j=0 \\ j \neq k}}^{2k} \binom{2k}{j} (-1)^{2k-j} \frac{(-1)^{j-k} - 1}{2(j-k)} + \binom{2k}{k} (-1)^k \frac{\pi}{2} \right]$$

$$= \frac{2(-1)^k}{(2i)^{2k}} \left[\underbrace{\sum_{\substack{j=0 \\ j-k \text{ impair}}}^{2k} \binom{2k}{j} \frac{1}{j-k}}_{\text{nul par symétrie}} + \binom{2k}{k} \frac{\pi}{2} \right]$$

$$= \binom{2k}{k} \frac{\pi}{2^{2k}} = \frac{\Gamma(k+\frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(k+1)} \quad !!$$